

Monotone-iterative technique for an initial value problem for difference equations with non-instantaneous impulses

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Abstract

In this paper a special type of difference equations is investigated. The impulses start abruptly at some points and their action continue on given finite intervals. This type of equations is used to model a real process. An algorithm, namely, the monotone iterative technique is suggested to solve the initial value problem for nonlinear difference equations with non-instantaneous impulses approximately. An important feature of our algorithm is that each successive approximation of the unknown solution is equal to the unique solution of an appropriately constructed initial value problem for a linear difference equation with with non-instantaneous impulses, and a formula for its explicit form is given. It is proved both sequences are convergent and their limits are minimal and maximal solutions of the considered problem.

Key words: difference equations, non-instantaneous impulses, lower and upper solutions, monotone-iterative technique.

2000 AMS subject classifications: 39A22, 65Q10

1 Introduction

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses:

- *instantaneous impulses*- the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, the monographs [5], [6], [9] and the cited references therein);

- *noninstantaneous impulses* - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan ([4]) introduced this new class of abstract differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions.

One of the problems in difference equations, which unknown function is involved in the present time on both parts of the equation nonlinearly, is the obtaining of the solution. Often it could be done in a closed form but approximately. One of the approximate method is based on the method of upper and lower solutions, combined with a monotone-iterative technique. It is used to construct two monotonous sequences of upper and lower solutions of the nonlinear non-instantaneous impulsive difference equation. This method is applied for difference equations in [8], [7] and for impulsive difference equations in [2].

The idea of the method is to use upper and lower solutions for initial iteration and to construct successive approximation from the corresponding non-instantaneous impulsive linear equation. These functional sequences converge monotonically to the minimal and maximal solutions of the nonlinear equation.

2 Statement of the problem

Let \mathbb{Z}_+ denote the set of all nonnegative integers. Let the increasing sequence $\{n_i\}_{i=0}^{p+1} : n_i \in \mathbb{Z}_+, n_i \geq n_{i-1} + 3, i = 1, 2, \dots, p$ and the sequence $\{d_i\}_{i=1}^p : d_i \in \mathbb{Z}_+, 1 \leq d_i \leq n_{i+1} - n_i - 2, i = 1, 2, \dots, p$ be given. We denote $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\}, a, b \in \mathbb{Z}_+, a < b$ and $I_k = \mathbb{Z}[n_k + d_k, n_{k+1} - 2], k \in \mathbb{Z}[0, p - 1], I_p = \mathbb{Z}[n_p + d_p, n_{p+1} - 1]$ and $J_k = \mathbb{Z}[n_k + 1, n_k + d_k], k \in \mathbb{Z}[1, p]$ where $d_0 = 0$.

Consider the *initial value problem (IVP)* for the nonlinear *non-instantaneous impulsive difference equation (NIDE)*

$$\begin{aligned} x(n+1) &= f(n, x(n), x(n+1)) \text{ for } n \in \bigcup_{k=0}^p I_k, \\ x(n_k) &= F(k, x(n_k - 1)), \quad k \in \mathbb{Z}[1, p] \\ x(n) &= g(n, x(n), x(n_k)) \text{ for } n \in \bigcup_{k=1}^p J_k, \\ x(n_0) &= x_0, \end{aligned} \tag{1}$$

where $x, x_0 \in \mathbb{R}, f : \bigcup_{k=0}^p I_k \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, F : \mathbb{Z}[1, p] \times \mathbb{R} \rightarrow \mathbb{R}$, and $g : \bigcup_{k=1}^p J_k \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

3 Preliminaries results

Definition 1. We will say that the function $\alpha : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R}$ is a minimal(maximal) solution of the IVP for NIDE (1) in $\mathbb{Z}[n_0, n_{p+1}]$ if it is a solution of (1) and for any solution $u(n), n \in \mathbb{Z}[n_0, n_{p+1}]$ the inequality $\alpha(n) \leq u(n)$ ($\alpha(n) \geq u(n)$) holds on $\mathbb{Z}[n_0, n_{p+1}]$

Definition 2. The function $\alpha : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R}$ is called lower (upper) solutions of IVP for NIDE (1), if:

$$\begin{aligned} \alpha(n+1) &\leq (\geq) f(n, \alpha(n), \alpha(n+1)), \text{ for } n \in \bigcup_{k=0}^p I_k, \\ \alpha(n_k) &\leq (\geq) F(k, \alpha(n_k - 1)), \text{ } k \in \mathbb{Z}[1, p] \\ \alpha(n) &\leq (\geq) g(n, \alpha(n), \alpha(n_k)), \text{ for } n \in \bigcup_{k=1}^p J_k \\ \alpha(n_0) &\leq (\geq) x_0 \end{aligned}$$

Consider the linear NIDE of the type

$$\begin{aligned} u(n+1) &= Q_n u(n) + \sigma_n, \text{ } n \in \bigcup_{k=0}^p I_k, \\ u(n_k) &= T_k u(n_k - 1) + \mu_k, \text{ } k \in \mathbb{Z}[1, p] \\ u(n) &= M_n u(n) + L_n u(n_k) + \gamma_n, \text{ } n \in \bigcup_{k=1}^p J_k \\ u(n_0) &= x_0, \end{aligned} \tag{2}$$

where $u, x_0 \in \mathbb{R}$, $Q_n : n \in \bigcup_{k=0}^p I_k$, $\sigma_n : n \in \bigcup_{k=0}^p I_k$, $L_n, M_n \neq 1, \gamma_n : n \in \bigcup_{k=1}^p J_k$, and $T_k, \mu_k : k \in \mathbb{Z}[1, p]$ are given real constants.

Lemma 1. The IVP for NIDE (2) has an unique solution given by

$$\begin{aligned} u(n) &= N(n) \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \sigma_j \prod_{i=j+1}^{n-1} Q_i + N(n) \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \prod_{i=j}^{n-1} Q_i \\ &\quad + \tau(n) \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}] \end{aligned} \tag{3}$$

where $\sigma_{n_0-1} = x_0$, $\sigma_n = 0$, $Q_n = 1$ for $n \in \mathbb{Z}[n_0, n_{p+1}] / \bigcup_{k=0}^p I_k$,

$$N(n) = \begin{cases} \frac{L_n}{1-M_n} & \text{for } n \in \bigcup_{k=1}^p J_k \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

$$\tau(n) = \begin{cases} \frac{\gamma_n}{1-M_n} & \text{for } n \in \bigcup_{k=1}^p J_k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$R(n) = \begin{cases} N(n_k + d_k) = \frac{L_{n_k+d_k}}{1-M_{n_k+d_k}} & \text{for } n = n_k + d_k + 1, \quad k \in \mathbb{Z}[1, p] \\ T_k & \text{for } n = n_k, \quad k \in \mathbb{Z}[1, p] \\ 1 & \text{otherwise} \end{cases} \quad (6)$$

$$\zeta(n) = \begin{cases} \tau(n_k + d_k) = \frac{\gamma_{n_k+d_k}}{1-M_{n_k+d_k}} & \text{for } n = n_k + d_k + 1, \quad k \in \mathbb{Z}[1, p] \\ \mu_k & \text{for } n = n_k, \quad k \in \mathbb{Z}[1, p] \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Proof: We will use an induction with respect to the interval. Let $n \in I_0 = \mathbb{Z}[n_0, n_1 - 2]$. Then we obtain $u(n) = \sum_{j=n_0-1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i$, $n \in \mathbb{Z}[n_0 + 1, n_1 - 1]$. Let $n = n_1$. Then using $\sigma_{n_1-1} = 0$, $Q_{n_1-1} = 1$ we get

$$\begin{aligned} u(n_1) &= T_1 u(n_1 - 1) + \mu_1 = T_1 \sum_{j=n_0-1}^{n_1-1} \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1 \\ &= \sum_{j=n_0-1}^{n_1-1} \left(\prod_{i=j+1}^{n_1} R(i) \right) \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1. \end{aligned}$$

Let $n \in J_1 = \mathbb{Z}[n_1 + 1, n_1 + d_1]$. Then using $\sigma_j = 0$, $Q_j = 1$, $j \in \mathbb{Z}[n_1, n_1 + d_1]$ we get

$$\begin{aligned} u(n) &= \frac{L_n}{1 - M_n} \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^{n_1} R(i) \right) \sigma_j \prod_{i=j+1}^{n-1} Q_i + \frac{L_n}{1 - M_n} \mu_1 + \frac{\gamma_n}{1 - M_n}. \\ &= N(n) \sum_{j=n_0-1}^{n-1} \sigma_j \left(\prod_{i=j+1}^n R(i) \right) \prod_{i=j+1}^{n-1} Q_i + N(n) \mu_1 + \tau(n), \quad n \in J_1. \end{aligned}$$

Let $n \in I_1 = \mathbb{Z}[n_1 + d_1, n_2 - 2]$. Then

$$\begin{aligned} u(n) &= N(n_1 + d_1) \sum_{j=n_0-1}^{n_1+d_1-1} \sigma_j \left(\prod_{i=j+1}^{n_1+d_1} R(i) \right) \prod_{i=j+1}^{n_1+d_1-1} Q_i \prod_{i=n_1+d_1}^{n-1} Q_i \\ &\quad + \mu_1 N(n_1 + d_1) \prod_{i=n_1+d_1}^{n-1} Q_i + \tau(n_1 + d_1) \prod_{i=n_1+d_1}^{n-1} Q_i \\ &\quad + \sum_{j=n_1+d_1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i \\ &= \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \sigma_j \prod_{i=j+1}^{n-1} Q_i + \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \prod_{i=j}^{n-1} Q_i. \end{aligned}$$

Let $n = n_2$. Then we get

$$\begin{aligned} u(n_2) &= T_1 u(n_2 - 1) + \mu_2 \\ &= \sum_{j=n_0-1}^{n_2-1} \left(\prod_{i=j+1}^{n_2} R(i) \right) \sigma_j \prod_{i=j+1}^{n_2-1} Q_i + \sum_{j=n_0}^{n_2} \left(\prod_{i=j+1}^{n_2} R(i) \right) \zeta(j) \prod_{i=j}^{n_2-1} Q_i. \end{aligned}$$

Let $n \in J_2 = \mathbb{Z}[n_2 + 1, n_2 + d_2]$. Then

$$\begin{aligned}
u(n) &= N(n)u(n_2) + \tau(n) \\
&= N(n) \sum_{j=n_0-1}^{n_2-1} \left(\prod_{i=j+1}^{n_2} R(i) \right) \sigma_j \prod_{i=j+1}^{n_2-1} Q_i + N(n) \sum_{j=n_0}^{n_2} \left(\prod_{i=j+1}^{n_2} R(i) \right) \zeta(j) \\
&\quad \times \prod_{i=j}^{n_2-1} Q_i + \tau(n) \\
&= N(n) \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \sigma_j \prod_{i=j+1}^{n-1} Q_i + N(n) \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \\
&\quad \times \prod_{i=j}^{n-1} Q_i + \tau(n).
\end{aligned}$$

Continue this process step by step w.r.t. the interval by induction we proves the solution of NIDE (2) is given by (3) for all $n \in \mathbb{Z}[n_0, n_{p+1}]$. \square

Lemma 2. Assume $m : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R}$ satisfies the inequalities

$$\begin{aligned}
m(n+1) &\leq Q_n m(n), \quad n \in \bigcup_{k=0}^p I_k, \\
m(n_k) &\leq T_k m(n_k - 1), \quad k \in \mathbb{Z}[1, p] \\
m(n) &\leq M_n m(n) + L_n m(n_k), \quad n \in \bigcup_{k=1}^p J_k \\
m(n_0) &\leq 0,
\end{aligned} \tag{8}$$

where $Q_n > 0$ ($n \in \bigcup_{k=0}^p I_k$), $T_k > 0$ ($k \in \mathbb{Z}[1, p]$) and $L_n > 0$, $M_n < 1$ ($n \in \bigcup_{k=0}^p J_k$).

Then $m(n) \leq 0$ for every $n \in \mathbb{Z}[n_0, n_{p+1}]$.

The proof is based on an induction w.r.t. the interval and we omit it.

4 Main results

For any pair of function $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R}$ such that $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$ we define the sets

$$S(\alpha, \beta) = \{u : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R} : \alpha(n) \leq u(n) \leq \beta(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]\}$$

$$\Omega_1(\alpha, \beta) = \{u \in \mathbb{R} : \alpha(n) \leq u \leq \beta(n), \quad n \in \bigcup_{k=0}^p I_k\}$$

$$\Omega_2(\alpha, \beta) = \{u \in \mathbb{R} : \alpha(n+1) \leq u \leq \beta(n+1), \quad n \in \bigcup_{k=0}^{p-1} I_k\}$$

$$\Lambda(\alpha, \beta) = \{u \in \mathbb{R} : \alpha(n) \leq u \leq \beta(n), \quad n \in \bigcup_{k=1}^p J_k\}$$

$$\Gamma(\alpha, \beta) = \{y \in \mathbb{R} : \alpha(n_k) \leq y \leq \beta(n_k), \quad k \in \mathbb{Z}[1, p]\}$$

$$\Upsilon(\alpha, \beta) = \{z \in \mathbb{R} : \alpha(n_k - 1) \leq z \leq \beta(n_k - 1), \quad k \in \mathbb{Z}[1, p]\}$$

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \rightarrow \mathbb{R}$ are lower and upper solutions of the IVP for NIDE (1) and $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.*
2. *The function $f : \bigcup_{k=0}^p I_k \times \Omega_1(\alpha, \beta) \times \Omega_2(\alpha, \beta) \rightarrow \mathbb{R}$ is continuous in its second and third arguments and there exist functions $K : \bigcup_{k=0}^p I_k \rightarrow (-\infty, 1)$ and $P : \bigcup_{k=0}^p I_k \rightarrow (0, \infty)$ such that for any $n \in \bigcup_{k=0}^p I_k$ and $x_1, x_2 \in \Omega_1(\alpha, \beta)$, with $x_1 \leq x_2$, and $x_3, x_4 \in \Omega_2(\alpha, \beta)$, with $x_3 \leq x_4$ the inequality*

$$f(n, x_1, x_3) - f(n, x_2, x_4) \leq P(n)(x_1 - x_2) + K(n)(x_3 - x_4)$$

holds.

3. *The function $F : \mathbb{Z}[1, p] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous in its second argument and there exists a function $T : \mathbb{Z}[1, p] \rightarrow (0, \infty)$ such that for any $k \in \mathbb{Z}[1, p]$ and $z_1, z_2 \in \Upsilon(\alpha, \beta)$ with $z_1 \leq z_2$*

$$F(k, z_1) - F(k, z_2) \leq T(k)(z_1 - z_2).$$

4. *The function $g : \bigcup_{k=1}^p J_k \times \Lambda(\alpha, \beta) \times \Gamma(\alpha, \beta) \rightarrow \mathbb{R}$ is continuous in its second and third arguments and there exist functions $M : \bigcup_{k=1}^p J_k \rightarrow (-\infty, 1)$ and $L : \bigcup_{k=1}^p J_k \rightarrow (0, \infty)$ such that for any $n \in \bigcup_{k=1}^p J_k$ and $y_1, y_2 \in \Lambda(\alpha, \beta)$, with $y_1 \leq y_2$, and $y_3, y_4 \in \Gamma(\alpha, \beta)$, with $y_3 \leq y_4$ the inequality*

$$g(n, y_1, y_3) - g(n, y_2, y_4) \leq M(n)(y_1 - y_2) + L(n)(y_3 - y_4)$$

holds.

Then there exist two sequences of discrete functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$, $n \in \mathbb{Z}[n_0, n_{p+1}]$ with $\alpha^{(0)} = \alpha$ and $\beta^{(0)} = \beta$ such that:

a) The sequences are nondecreasing and nonincreasing, respectively and

$$\alpha(n) \leq \alpha^{(j)}(n) \leq \beta^{(j)}(n) \leq \beta(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}];$$

b) The functions $\alpha^{(j)}(n)$ and $\beta^{(j)}(n)$ are lower and upper solutions of the IVP for NIDE (1), respectively;

c) Both sequences are convergent on $\mathbb{Z}[n_0, n_{p+1}]$;

d) The limits $\lim_{j \rightarrow \infty} \alpha^{(j)}(n) = A(n)$, $\lim_{j \rightarrow \infty} \beta^{(j)}(n) = B(n)$ are the minimal and maximal solutions of IVP for NIDE (1) in $S(\alpha, \beta)$, respectively;

e) If IVP for NIDE (1) has an unique solution $u(n) \in S(\alpha, \beta)$, then $A(n) \equiv u(n) \equiv B(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Proof: For any arbitrary fixed function $\eta \in S(\alpha, \beta)$, we consider the IVP for the linear NIDE

$$\begin{aligned} u(n+1) &= P(n)u(n) + K(n)u(n+1) + \psi(n, \eta(n), \eta(n+1)), \quad n \in \bigcup_{k=0}^p I_k \\ u(n_k) &= T(k)u(n_k - 1) + v(k, \eta(n_k - 1)), \quad k \in \mathbb{Z}[1, p] \\ u(n) &= M(n)u(n) + L(n)u(n_k) + \xi(n, \eta(n), \eta(n_k)), \quad n \in \bigcup_{k=1}^p J_k \\ u(n_0) &= x_0, \end{aligned} \tag{9}$$

where $u, x_0 \in \mathbb{R}$, and

$$\begin{aligned} \psi(n, x, y) &= f(n, x, y) - P(n)x - K(n)y, \quad n \in \bigcup_{k=0}^p I_k, \\ v(k, x) &= F(k, x) - T(k)x, \quad k \in \mathbb{Z}[1, p] \\ \xi(n, x, y) &= g(n, x, y) - M(n)x - L(n)y, \quad n \in \bigcup_{k=1}^p J_k. \end{aligned} \tag{10}$$

According to Lemma 1 the IVP for linear NIDE (9) has an unique solution given by (3) with $\sigma_n = \frac{\psi(n, \eta(n), \eta(n+1))}{1-K(n)}$, $Q_n = \frac{P(n)}{1-K(n)}$, $\gamma_n = \xi(n, \eta(n), \eta(n_j))$, $\mu_k = v(k, \eta(n_k - 1))$, $T_k = T(k)$, $M_n = M(n)$, $L_n = L(n)$.

For any function $\eta \in S(\alpha, \beta)$ we define the operator $Q : S(\alpha, \beta) \rightarrow S(\alpha, \beta)$ by $Q\eta = u$, where u is the unique solution of IVP for the linear NIDE (9) for the function η . The operator Q has the following properties:

(P1) $\alpha \leq Q\alpha$, $\beta \geq Q\beta$

(P2) Q is a monotone nondecreasing operator in $S(\alpha, \beta)$.

To prove (P1) set $Q\alpha = \alpha^{(1)}$, where $\alpha^{(1)}$ is the unique solution of (9) with $\eta = \alpha$ and let $m(n) = \alpha(n) - \alpha^{(1)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$

For any $n \in \bigcup_{k=0}^p I_k$ we obtain the inequality

$$\begin{aligned} m(n+1) &= \alpha(n+1) - P(n)\alpha^{(1)}(n) - K(n)\alpha^{(1)}(n+1) \\ &\quad - \psi(n, \alpha(n), \alpha(n+1)) \\ &\leq P(n)(\alpha(n) - \alpha^{(1)}(n)) + K(n)(\alpha(n+1) - \alpha^{(1)}(n+1)) \\ &= P(n)m(n) + K(n)m(n+1). \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)} m(n)$ holds for $n \in \bigcup_{k=0}^p I_k$.

For any $n = n_k$ we obtain

$$\begin{aligned} m(n_k) &\leq F(k, \alpha(n_k - 1)) - T(k)\alpha^{(1)}(n_k - 1) - F(k, \alpha(n_k - 1)) \\ &\quad + T(k)\alpha(n_k - 1) = T(k)m(n_k - 1). \end{aligned}$$

For any $n \in \bigcup_{k=1}^p J_k$ we get

$$\begin{aligned} m(n) &\leq g(n, \alpha(n), \alpha(n_k)) - M(n)\alpha^{(1)}(n) - L(n)\alpha^{(1)}(n_k) \\ &\quad - g(n, \alpha(n), \alpha(n_k)) + M(n)\alpha(n) + L(n)\alpha(n_k) \\ &= M(n)m(n) + L(n)m(n_k) \end{aligned}$$

Therefore, the function $m(n)$ satisfies the inequalities (8) with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k)$, $M_n = M(n)$, $L_n = L(n)$. According to Lemma 2 the function $m(n)$ is non-positive in $\mathbb{Z}[n_0, n_{p+1}]$, i.e. $\alpha \leq Q\alpha$. Analogously it can be proved that the inequality $\beta \geq Q\beta$ holds.

To prove (P2) we consider two arbitrary function $\eta_1, \eta_2 \in S(\alpha, \beta)$ such that $\eta_1(n) \leq \eta_2(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$. Let $u^{(1)} = Q\eta_1$ and $u^{(2)} = Q\eta_2$. Denote $m(n) = u^{(1)}(n) - u^{(2)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^p I_k$ we obtain the inequality

$$\begin{aligned} m(n+1) &= P(n)u^{(1)}(n) + K(n)u^{(1)}(n+1) + f(n, \eta_1(n), \eta_1(n+1)) \\ &\quad - P(n)\eta_1(n) - K(n)\eta_1(n+1) - P(n)u^{(2)}(n) - K(n)u^{(2)}(n+1) \\ &\quad - f(n, \eta_2(n), \eta_2(n+1)) + P(n)\eta_2(n) + K(n)\eta_2(n+1) \\ &\leq P(n)m(n) + K(n)m(n+1) \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)} m(n)$ holds for $n \in I_k, k \in \mathbb{Z}[0, p]$.

For any $n = n_k, k \in \mathbb{Z}[1, p]$ we get

$$\begin{aligned} m(n_k) &= T(k, u^{(1)}(n_k - 1) - u^{(2)}(n_k - 1)) - T(k)(\eta_1(n_k - 1) - \eta_2(n_k - 1)) \\ &\quad + F(k, \eta_1(n_k - 1)) - F(k, \eta_2(n_k - 1)) \leq T(k)m(n_k - 1) \end{aligned}$$

For any $n \in \bigcup_{k=1}^p J_k$ we obtain

$$\begin{aligned} m(n) &= M(n)u^{(1)}(n) + L(n)u^{(1)}(n_k) + g(n, \eta_1(n), \eta_1(n_k)) \\ &\quad - M(n)\eta_1(n) - L(n)\eta_1(n_k) - M(n)u^{(2)}(n) - L(n)u^{(2)}(n_k) \\ &\quad - g(n, \eta_2(n), \eta_2(n_k)) + M(n)\eta_2(n) + L(n)\eta_2(n_k) \\ &\leq M(n)m(n) + L(n)m(n_k) \end{aligned}$$

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k)$, $M_n = M(n)$, $L_n = L(n)$ the function $m(n) \leq 0$, i.e. $Q\eta_1 \leq Q\eta_2$, for $\eta_1(n) \leq \eta_2(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Let $\eta \in S(\alpha, \beta)$ be a lower solution of (1). We consider the function $Q\eta = m$. According to the proved $\eta(n) \leq m(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^p I_k$ we get the inequality

$$\begin{aligned} m(n+1) &= P(n)m(n) + K(n)m(n+1) + f(n, \eta(n), \eta(n+1)) \\ &\quad - P(n)\eta(n) - K(n)\eta(n+1) \\ &\leq f(n, m(n), m(n+1)) \end{aligned} \quad (11)$$

For any $n = n_k, k \in \mathbb{Z}[1, p]$ we obtain

$$\begin{aligned} m(n_k) &= F(k, m(n_k - 1)) - F(k, m(n_k - 1)) + T(k)m(n_k - 1) \\ &\quad + F(k, \eta(n_k - 1)) - T(k)\eta(n_k - 1) \\ &\leq F(k, m(n_k - 1)) \end{aligned} \quad (12)$$

For any $n \in \bigcup_{k=1}^p J_k$ we obtain

$$\begin{aligned} m(n) &= g(n, m(n), m(n_k)) - g(n, m(n), m(n_k)) + M(n)m(n) \\ &\quad + L(n)m(n_k) + f(n, \eta(n), \eta(n_k)) - M(n)\eta(n) - L(n)\eta(n_k) \\ &\leq g(n, m(n), m(n_k)) \end{aligned} \quad (13)$$

Inequalities (11), (12) and (13) prove the function m is a lower solution of NIDE (1). Similarly, if $\eta \in S(\alpha, \beta)$ is an upper solution of NIDE (1) then the function $m = Q\eta$ is an upper solution of (1).

We define the sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$ by the equalities $\alpha^{(0)} = \alpha$, $\beta^{(0)} = \beta$, $\alpha^{(j)} = Q\alpha^{(j-1)}$, $\beta^{(j)} = Q\beta^{(j-1)}$. The functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ satisfy the initial value problem (9) with $\eta(n) = \alpha^{(s-1)}(n)$ and $\eta(n) = \beta^{(s-1)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$, respectively.

According to Lemma 2 the following representations are valid:

$$\begin{aligned} \alpha^{(s)}(n) &= N(n) \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \frac{\psi(j, \alpha^{(s-1)}(j), \alpha^{(s-1)}(j+1))}{1 - K(j)} \\ &\quad \times \prod_{i=j+1}^{n-1} \frac{P(i)}{1 - K(i)} + \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1 - K(i)} + \tau(n), \end{aligned} \quad (14)$$

for $n \in \mathbb{Z}[n_0, n_{p+1}]$,

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, \alpha^{(s-1)}(n), \alpha^{(s-1)}(n_j))$, $n \in \bigcup_{k=1}^p J_k$, $j \in \mathbb{Z}[1, p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k+d_k, \alpha^{(s-1)}(n_k+d_k), \alpha^{(s-1)}(n_k))$, $\mu_k =$

$v(k, \alpha^{(s-1)}(n_k - 1)), k \in \mathbb{Z}[1, p]$.

$$\begin{aligned} \beta^{(s)}(n) &= N(n) \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \frac{\psi(j, \beta^{(s-1)}(j), \beta^{(s-1)}(j+1))}{1 - K(j)} \\ &\times \prod_{i=j+1}^{n-1} \frac{P(i)}{1 - K(i)} + \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1 - K(i)} + \tau(n), \end{aligned} \quad (15)$$

for $n \in \mathbb{Z}[n_0, n_{p+1}]$,

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, \beta^{(s-1)}(n), \beta^{(s-1)}(n_j))$, $j \in \mathbb{Z}[1, p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k + d_k, \beta^{(s-1)}(n_k + d_k), \beta^{(s-1)}(n_k))$, $\mu_k = v(k, \beta^{(s-1)}(n_k - 1))$, $k \in \mathbb{Z}[1, p]$.

According to the above proved, functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ are lower and upper solutions of NIDE (1), respectively and they satisfy for $n \in \mathbb{Z}[n_0, n_{p+1}]$ the following inequalities

$$\alpha^{(0)}(n) \leq \alpha^{(1)}(n) \leq \dots \leq \alpha^{(s)}(n) \leq \beta^{(s)}(n) \leq \dots \leq \beta^{(1)}(n) \leq \beta^{(0)}(n) \quad (16)$$

Both sequences of discrete functions being monotonic and bounded are convergent on $\mathbb{Z}[n_0, n_{p+1}]$.

Let $A(n) = \lim_{s \rightarrow \infty} \alpha^{(s)}(n)$, $B(n) = \lim_{s \rightarrow \infty} \beta^{(s)}(n)$.

Take a limit in (14) for $s \rightarrow \infty$ we obtain (3) with $u(n) = A(n)$, $\sigma_j = \frac{\psi(j, A(j), A(j+1))}{1 - K(j)}$, $Q_i = \frac{P(i)}{1 - K(i)}$,

$$\begin{aligned} A(n) &= N(n) \sum_{j=n_0-1}^{n-1} \left(\prod_{i=j+1}^n R(i) \right) \frac{\psi(j, A(j), A(j+1))}{1 - K(j)} \prod_{i=j+1}^{n-1} \frac{P(i)}{1 - K(i)} \\ &+ \sum_{j=n_0}^n \left(\prod_{i=j+1}^n R(i) \right) \zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1 - K(i)} + \tau(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}], \end{aligned} \quad (17)$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, A(n), A(n_j))$, $j \in \mathbb{Z}[1, p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k + d_k, A(n_k + d_k), A(n_k))$, $\mu_k = v(k, A(n_k - 1))$, $k \in \mathbb{Z}[1, p]$.

From (17) it follows the function $A(n)$ is a solution of NIDE (1).

Similarly, we prove the function $B(n)$ is a solution of NIDE (1).

Let $u \in S(\alpha, \beta)$ be a solution of IVP for NIDE (1). From inequalities (16) it follows there exists a natural number p such that $p \in \mathbb{N}$:

$$\alpha^{(p)}(n) \leq u(n) \leq \beta^{(p)}(n) \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}].$$

We introduce the notation $m(n) = \alpha^{(p+1)}(n) - u(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^p I_k$ we get the inequality

$$\begin{aligned} m(n+1) &= P(n)\alpha^{(p+1)}(n) + K(n)\alpha^{(p+1)}(n+1) + f(n, \alpha^{(p)}(n), \alpha^{(p)}(n+1)) \\ &\quad - P(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n+1) - f(n, u(n), u(n+1)) \\ &\leq P(n)m(n) + K(n)m(n+1) \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)} m(n)$ holds for $n \in \bigcup_{k=0}^p I_k$.

For any $n = n_k$, $k \in \mathbb{Z}[1, p]$ we obtain

$$\begin{aligned} m(n_k) &= T(k)\alpha^{(p+1)}(n_k - 1) + T(k)u(n_k - 1) - T(k)u(n_k - 1) \\ &\quad + F(k, (\alpha^{(p)}(n_k - 1)) - T(k)\alpha^{(p)}(n_k - 1) - F(k, u(n_k - 1)) \\ &\leq T(k)m(n_k - 1) \end{aligned}$$

For any $n \in \bigcup_{k=1}^p J_k$ we obtain

$$\begin{aligned} m(n) &= M(n)\alpha^{(p+1)}(n) + L(n)\alpha^{(p+1)}(n_k) + g(n, \alpha^{(p)}(n), \alpha^{(p)}(n_k)) \\ &\quad - M(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n_k) - g(n, u(n), u(n_k)) \\ &\leq M(n)m(n) + L(n)m(n_k) \end{aligned}$$

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k)$, $M_n = M(n)$, $L_n = L(n)$ the function $m(n)$ is nonpositive, i.e. $\alpha^{(p+1)}(n) \leq u(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$. Similarly $\beta^{(p+1)}(n) \geq u(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$, and hence $\alpha^{(j+1)} \leq u(n) \leq \beta^{(j+1)}$, $n \in \mathbb{Z}[n_0, n_{p+1}]$. Since $\alpha^{(0)}(n) \leq u(n) \leq \beta^{(0)}(n)$ this proves by induction that $\alpha^{(j)}(n) \leq u(n) \leq \beta^{(j)}(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$, for every j .

Taking the limit as $j \rightarrow \infty$ we conclude $A(n) \leq u(n) \leq B(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$. Hence $A(n)$ and $B(n)$ are minimal and maximal solutions of IVP for NIDE (1), respectively.

Let the IVP for NIDE (1) has an unique solution $u(n) \in S(\alpha, \beta)$.

Then from above it follows $A(n) \equiv u(n) \equiv B(n)$, $n \in \mathbb{Z}[n_0, n_{p+1}]$. \square

5 Conclusions

An algorithm for approximate solving an initial value problem for a nonlinear difference equations with non-instantaneous impulses is given and theoretically studied. It is based on the application of the method of lower and upper solutions. The suggested algorithm is appropriate for computerized and easy application to study discrete dynamical models.

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